

THE MOTIVE OF THE MODULI STACK OF G -BUNDLES OVER THE UNIVERSAL CURVE

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ABSTRACT. We define relative motives in the sense of André. After associating a complex in the derived category of motives to an algebraic stack we study this complex in the case of the moduli of G -bundles varying over the moduli of curves.

For a classical group G , let $\mathrm{Bun}_{G,C}^s$ denote the moduli space of stable G -bundles over a smooth complex projective curve C . As C varies over the space of complex genus g curves $M_g(\mathbb{C})$, the rational cohomology $H^i(\mathrm{Bun}_{G,C}^s, \mathbb{Q})$ fits together to form a variation of mixed Hodge structure. One of the aims of this work is to understand this variation of mixed Hodge structure. The behaviour for large i seems rather subtle, and will be left for the future. In this paper, we determine this structure completely for i less than an explicit constant depending on G and g . The main point is that in this range, the cohomology of $\mathrm{Bun}_{G,C}^s$ agrees with the cohomology of the moduli stack $\mathfrak{Bun}_{G,C}$ of all G -bundles on C (section 5.7). Perhaps contrary to one's first impression, the stack turns out to be the more accessible of the two objects for this problem. We show that for all i , the variation of mixed Hodge structure associated to $H^i(\mathfrak{Bun}_{G,C})$ can be built out of the tautological variation of pure Hodge structure associated to $H^1(C, \mathbb{Q})$ using standard linear algebra operations and Tate twists (section 4.6); in particular, it is pure and the Torelli group acts trivially.

The above statements are deduced from finer results at the motivic level. Motives come in different flavours, and in this paper we present yet another, which is a relative theory of pure motives over a base \mathfrak{S} . The base is allowed to be a quotient of a smooth variety defined over a subfield of \mathbb{C} by a finite group. When \mathfrak{S} is a point, the theory reduces to André's [An]. In general, the category of motives $M_A(\mathfrak{S})$ in the present sense, over \mathfrak{S} , forms a semisimple Tannakian category. There are realizations of $M_A(\mathfrak{S})$ to the category of ℓ -adic sheaves over \mathfrak{S} and variations of Hodge structure over $\mathfrak{S} \times \mathrm{Spec} \mathbb{C}$. To every Artin stack \mathfrak{X}/\mathbb{C} , we can associate a well defined motive $h^i(\mathfrak{X})$, which maps to the pure Hodge structure $Gr_W H^i(\mathfrak{X})$ under the Hodge realization. Similar statements hold in the relative case. The main theorem (theorem 4.8) is a motivic version of the Atiyah-Bott isomorphism, which gives a precise description of the motive of the stack of G -bundles over the universal curve relative to the moduli stack of all curves.

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1. STACKS

By a stack \mathfrak{X} , we will mean an algebraic stack in Artin's sense, which is locally of finite type over some base scheme. We give a working definition which is sufficient for our needs, but refer to [LM] for the full story. Given a groupoid $(s, t : R \rightrightarrows U, \dots)$ in the category of algebraic spaces over some base, such that s, t are surjective and smooth (or étale in the special case of Deligne-Mumford stacks), we can associate a stack called the quotient stack $\mathfrak{X} = [U/R]$, which we can think of as an equivalence class in a sense to be explained. Given a surjective smooth morphism $V \rightarrow U$, we get a new groupoid $(R_V = R \times_{U \times U} (V \times V) \rightrightarrows V, \dots)$ by base change. We define isomorphism for the quotients as the weakest equivalence relation such that $[U/R] \cong [V/R_V]$ for all such base changes. In the trivial case, where s and t are the identity, we just get back U . So stacks include algebraic spaces. Stacks form a category (actually a 2-category), where a morphism (or more accurately a 1-morphism) is given by a morphism between some pair of defining groupoids. Given a stack \mathfrak{X} , we get a groupoid valued functor $Y \mapsto \mathfrak{X}(Y) = \text{Hom}(Y, \mathfrak{X})$ on the category of algebraic spaces. This functor determines \mathfrak{X} and typically gives the most natural description of it. In cases of interest to us, the groupoid comes from an action of a group G on U ; so $R = G \times U$ with s, t being respectively the projection and action maps. If the quotient U/G exists as an algebraic space, we get a morphism $[U/G] \rightarrow U/G$ which need not be an isomorphism. The quotient U/G is usually referred to as the coarse moduli space associated to $[U/G]$, and it can be characterized by the fact that any morphism of $[U/G]$ to an algebraic space would factor through it uniquely.

We list some key examples along with the functors they represent.

Example 1.1. Let G be a group scheme over a field κ . The classifying stack $BG = [\text{Spec } \kappa/G]$. The universal bundle $EG = [G/G] = *$ maps to BG . The quotient scheme $\text{Spec } \kappa/G$, is trivial, while BG is not. $BG(Y)$ is just the groupoid of principal bundles on Y and isomorphisms between them.

Example 1.2. Let $\mathfrak{M}_g/\text{Spec } \mathbb{Q}$ be the moduli stack of smooth projective curves of genus g . This is the quotient (stack) of the Hilbert scheme of tricanonically embedded curves H_g by the appropriate projective linear group PGL . We can also realize this as the quotient stack of the fine moduli space $M_{g,n}$ of curves with level $n \geq 3$ structure by $\text{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. The quotient of the universal curve $C_g \rightarrow H_g$ by PGL group yields a morphism of stacks $\mathfrak{C}_g \rightarrow \mathfrak{M}_g$. $\mathfrak{M}_g(Y)$ is the groupoid of genus g curves over Y and $\mathfrak{C}_g(Y)$ is the groupoid of curves with a distinguished section. The coarse moduli space for \mathfrak{M}_g is just the moduli space M_g .

Example 1.3. Given an algebraic group G over a field κ and a κ -scheme S . Let $\mathfrak{Bun}_{G,X/S}$ be the moduli stack of principal G -bundles over a flat family X/S , so that $\mathfrak{Bun}_{G,X/S}(Y)$ is the groupoid of G -bundles over $X \times_S Y$. There is morphism of stacks $UG \rightarrow X \times \mathfrak{Bun}_G(X)$ which serves as the universal G -bundle. When $X = \text{Spec } \kappa$, this is the classifying stack BG and $UG = EG$. There is an action of PGL on $\mathfrak{Bun}_{G,C_g/H_g}$ and by passage to quotients we obtain

$$\pi : \mathfrak{Bun}_{G,\mathfrak{C}_g/\mathfrak{M}_g} \rightarrow \mathfrak{M}_g$$

which is the object of fundamental interest here.

The details of the construction of $\mathfrak{Bun}_G = \mathfrak{Bun}_{G,X/S}$ can be found in [LM] for vector bundles ($G = GL$), and we can reduce the general case to this. Choose a

faithful representation $G \hookrightarrow GL$, then the fibre of the natural map

$$\rho : \mathfrak{Bun}_G \rightarrow \mathfrak{Bun}_{GL}$$

over a principal GL -bundle E is the set of all reductions of the structure group of E to G , or in other words sections of $E/G \rightarrow X$. Since E/G is a quasiprojective variety over X , its sections are representable by a subscheme of a Hilbert scheme. This argument carried out in families shows that the morphism ρ is representable, which implies that \mathfrak{Bun}_G is an algebraic stack.

If X is a smooth projective curve over a scheme S and \mathcal{G} is a smooth group scheme over X then $\mathfrak{Bun}_{\mathcal{G}}$ is also an algebraic stack. This requires more work and the details can be found in [Be1].

A groupoid can be extended to a simplicial algebraic space called its nerve

$$\dots R \times_{s,U,t} R \times_{s,U,t} R \xrightarrow{\sim} R \rightrightarrows U$$

When working over \mathbb{C} , the geometric realization of the corresponding analytic object gives a topological space $|[U/R]|$ whose weak homotopy type depends only on the underlying stack. Thus we have well defined notions of singular cohomology and fundamental groups for stacks over \mathbb{C} . This can be refined to show that stacks carry natural mixed Hodge structures on cohomology (of possibly infinite dimension). See [Dh, T] for details. More generally, given a stack $F : \mathfrak{X} \rightarrow S$ over a complex base scheme, we can define direct images $R^i F_* \mathbb{Q}$ by the same procedure. We record the following lemma which is straightforward.

Lemma 1.4. *If X is a connected variety on which a finite group G acts, and $\mathfrak{X} = [X/G]$. There is an exact sequence of fundamental groups*

$$1 \rightarrow \pi_1(X) \rightarrow \pi_1(\mathfrak{X}) \rightarrow G \rightarrow 1$$

and $\pi_1(\mathfrak{X})$ is determined by the action of G on the fundamental groupoid of X .

When $\mathfrak{X} = BG$, the geometric realization of the nerve is just the usual bar construction for the classifying space. And for $\mathfrak{X} = [U/G]$, this is nothing but the homotopy quotient $(U \times EG)/G$. When $\mathfrak{X} = \mathfrak{M}_g$, the rational cohomology is the same as for the moduli space M_g . However, the fundamental groups are different. When $X = M_{g,n}$, the sequence in the lemma is

$$1 \rightarrow \Gamma_{g,n} \rightarrow \Gamma_g \rightarrow Sp(2g, \mathbb{Z}/n\mathbb{Z}) \rightarrow 1$$

where Γ_g is the mapping class group, and $\Gamma_{g,n}$ is its n th congruence subgroup.

2. RELATIVE MOTIVES

Relative Chow and homological motives have been constructed by Denninger-Murre [DM] and Corti-Hanamura [CH] respectively. Our goal is to define a relative version of André's category of motives, whose construction we now recall [An]. Let κ be a field of characteristic 0 embeddable into \mathbb{C} with algebraic closure $\bar{\kappa}$. We will fix this notation for the remainder of the paper. Let $SPVar_{\kappa}$ be the category of smooth projective (possibly reducible) varieties over κ . Fix the Weil cohomology $H^*(X) = H^*(X_{et}, \mathbb{Q}_{\ell})$ for the moment. André has constructed a \mathbb{Q} -subalgebra $A_{mot}^*(X) \subset H^{2*}(X)$ of motivated cycles on X . A class $\gamma \in A_{mot}^*(X)$ if and only if there exists an object $Y \in SPVar_{\kappa}$ and algebraic cycles α, β on $X \times Y$ such that

$$\gamma = p_*(\alpha \cup * \beta),$$

where $p : X \times Y \rightarrow X$ is the projection, and $*$ is the Lefschetz involution with respect to a product polarization [An]. Given a second Weil cohomology H'^* and comparison isomorphisms $H^* \otimes K \cong H'^* \otimes K$, where K is a common overfield of the coefficient fields, we can identify motivated cycles with respect to H^* and H'^* . Thus the initial choice of Weil cohomology is immaterial. Choosing H'^* to be Betti i.e. singular cohomology for $X \times_{\sigma} \text{Spec } \mathbb{C}$ (for an embedding $\sigma : \kappa \rightarrow \mathbb{C}$), we find that motivated cycles map to Hodge cycles. By varying ℓ and σ , we see that motivated cycles map to absolute Hodge cycles.

The category M_A of André motives, can now be constructed by following the standard procedure of first constructing a category whose objects are the same as for $SPVar_{\kappa}$, but with $\text{Hom}(X, Y)$ given correspondences $A_{\text{mot}}^{\dim X}(X, Y)$, and then taking the pseudo-abelian completion and then inverting the Lefschetz motive. Or it can be defined in one step à la Jannsen [An, sect. 4]. The nice feature is that M_A is semisimple \mathbb{Q} -linear abelian category; this is only conjecturally true for homological motives. Let $H^*(X)$ denote ℓ -adic cohomology with its $\text{Gal}(\kappa)$ -action or rational singular cohomology of $X \times_{\sigma} \text{Spec } \mathbb{C}$, with its canonical Hodge structure. Then each smooth projective variety has a functorial motive $h(X)$, such that the functor $X \mapsto H(X) = \bigoplus_i H^i(X)$ factors through it. This yields a faithful embedding of M_A into the category of ℓ -adic representations or Hodge structures. Moreover, since A_{mot} contains Künneth projections, we can decompose $h(X) = \bigoplus_i h^i(X)$ in M_A such that the realization of $h^i(X)$ is $H^i(X)$. We note also that $A_{\text{mot}}^i(X) = \text{Hom}(\mathbb{Q}(-i), h(X)) = \text{Hom}(\mathbb{Q}(-i), h^{2i}(X))$. Define a motive to have weight i if it isomorphic to summand of $h^j(X)(k)$ for some X and j, k with $i = j - 2k$. Any motive T decomposes canonically into a direct sum of pure motives $w^i(T)$ of weight i . Under Hodge realization, $w^i(T)$ corresponds to the maximal sub Hodge structure of weight i .

Let S be a geometrically connected smooth variety over κ with an action by a finite group G . Set $\mathfrak{S} = [S/G]$. (It may be helpful to the following examples in mind: $G = \{1\}$ so $\mathfrak{S} = S$, or $S = M_{g,n}$ and $\mathfrak{S} = \mathfrak{M}_g$.) Let $SPVar_{\mathfrak{S}}$ be the category of representable smooth projective morphisms to \mathfrak{S} . Any object of this category $\mathfrak{F} : \mathfrak{X} \rightarrow \mathfrak{S}$ can be pulled back to a G -equivariant smooth projective morphism $f : X \rightarrow S$ and conversely. We will keep this notation throughout this section. To every $\mathfrak{F} : \mathfrak{X} \rightarrow \mathfrak{S}$ in $SPVar_{\mathfrak{S}}$, we will define a motive $h^0(\mathfrak{S}, R^i \mathfrak{F}_* \mathbb{Q}) \in M_A$ such that its Betti realization gives $H^0(S \otimes \mathbb{C}, R^i f_* \mathbb{Q})^G$ for every embedding $\kappa \subset \mathbb{C}$. Choose a G -equivariant nonsingular compactification \bar{X} and a base point $s \in S(\kappa)$. Then G will act on $h^i(\bar{X})$. Set

$$\begin{aligned} h^0(\mathfrak{S}, R^i \mathfrak{F}_* \mathbb{Q}) &= \text{im}[h^i(\bar{X})^G \rightarrow h^i(X_s)] \\ &\cong \frac{h^i(\bar{X})^G}{\ker(h^i(\bar{X}) \rightarrow h^i(X_s)) \cap h^i(\bar{X})^G} \end{aligned}$$

and

$$h^0(\mathfrak{S}, \mathbb{R} \mathfrak{F}_* \mathbb{Q}) = \bigoplus_i h^0(\mathfrak{S}, R^i \mathfrak{F}_* \mathbb{Q})$$

The arguments of [An, p. 25] shows that these are independent of choices. When $S = \mathfrak{S}$, [Ar1, 2.6] would in fact imply the existence of well defined higher Leray motives $h^j(S, R^i f_* \mathbb{Q})$. This construction can be extended to the general case, but we will skip the details since h^0 is sufficient for the needs of this paper.

We define

$$A_{mot}^i(\mathfrak{X}/\mathfrak{S}) = \text{Hom}(\mathbb{Q}(-i), h^0(\mathfrak{S}, R^{2i}\mathfrak{F}_*\mathbb{Q}))$$

The next proposition will give some useful alternative descriptions, when $\kappa = \mathbb{C}$. We note that $R^j\mathfrak{F}_*\mathbb{Q}$ can be pulled back to a local system on $(S \times EG)/G$, thus $\pi_1(\mathfrak{S})$ acts on the fibre $H^j(X_s)$.

Proposition 2.1. *Assume $\kappa = \mathbb{C}$ and let H^* denote rational singular cohomology. With \bar{X} and s as above, we have*

$$A_{mot}^i(\mathfrak{X}/\mathfrak{S}) = \text{im}(A_{mot}^i(\bar{X})^G \rightarrow H^{2i}(X_s)) = A_{mot}^i(X_s) \cap H^{2i}(X_s)^{\pi_1(\mathfrak{S})}$$

Proof. For the first part note that

$$\begin{aligned} A_{mot}^i(\mathfrak{X}/\mathfrak{S}) &= \text{Hom}(\mathbb{Q}(-i), h^0(S, R^{2i}\mathfrak{F}_*\mathbb{Q})) \\ &= \text{Hom}(\mathbb{Q}(-i), \text{im}[h^{2i}(\bar{X})^G \rightarrow h^{2i}(X_s)]) \\ &= \text{im}[\text{Hom}(\mathbb{Q}(-i), h^{2i}(\bar{X}))^G \rightarrow \text{Hom}(\mathbb{Q}(-i), h^{2i}(X_s))] \\ &= \text{im}[A_{mot}^i(\bar{X})^G \rightarrow A_{mot}^i(X_s)] \\ &= \text{im}[A_{mot}^i(\bar{X})^G \rightarrow H^{2i}(X_s)] \end{aligned}$$

The last equality follows from the functoriality of motivated cycles under restriction with respect to $X_s \subset \bar{X}$.

From lemma 1.4, we see that $H^{2i}(X_s)^{\pi_1(\mathfrak{S})} = (H^{2i}(X_s)^{\pi_1(S)})^G$. Thus we have

$$\begin{aligned} \text{im}[A_{mot}^i(\bar{X})^G \rightarrow H^{2i}(X_s)] &\subseteq A_{mot}^i(X_s) \cap (H^{2i}(X_s)^{\pi_1(S)})^G \\ &= A_{mot}^i(X_s) \cap H^{2i}(X_s)^{\pi_1(\mathfrak{S})} \end{aligned}$$

So it suffices to prove the reverse inclusion. Suppose that $\xi_s \in A_{mot}^i(X_s) \cap H^{2i}(X_s)^{\pi_1(\mathfrak{S})}$. Since it is both $\pi_1(S)$ and G -invariant, it can be transported to a G -invariant cycle ξ_t on any fibre X_t . By [An, thm 0.5], this is again motivated. So we can express $\xi_t = p_*(\alpha_t \cup * \beta_t)$ for algebraic cycles α_t, β_t on $X_t \times Y$ for some Y . By taking t general, we can (by standard Hilbert scheme arguments) assume that these cycles extend to algebraic cycles α, β on $f^{-1}(U) \times Y$ for some Zariski neighbourhood of t . Then by taking closures, we get algebraic cycles $\bar{\alpha}, \bar{\beta}$ on $\bar{X} \times Y$ such that

$$\xi_s = p_*(\bar{\alpha} \cup * \bar{\beta})|_{X_s} \in \text{im } A_{mot}^i(\bar{X})$$

Furthermore by averaging over the group, we can assume these cycles are G -invariant. \square

By Artin's comparison theorem, we obtain

Corollary 2.2.

$$A_{mot}^i(\mathfrak{X}/\mathfrak{S}) = A_{mot}^i(X_s) \cap H^0(S_{et}, R^{2i}f_*\mathbb{Q}_\ell)^G$$

Define the category $\text{Cor}_{A_{mot}}(\mathfrak{S})$ of relative motivated correspondences having as objects smooth projective morphisms, and

$$\text{Hom}(\mathfrak{X}/\mathfrak{S}, \mathfrak{Y}/\mathfrak{S}) = \prod_i A_{mot}^{\dim X_i - \dim S}(\mathfrak{X}_i \times_{\mathfrak{S}} \mathfrak{Y}),$$

where X_i are the connected components of $X = \mathfrak{X} \times_{\mathfrak{S}} S$. Composition is defined by the usual rule [K].

The category $M_A(\mathfrak{S})^{st}$ of strict motives over \mathfrak{S} can be constructed from the category $Cor_{A_{mot}}(\mathfrak{S})$ as above. Each smooth projective map $\mathfrak{F} : \mathfrak{X} \rightarrow \mathfrak{S}$ gives rise to a motive $h(\mathfrak{X}/\mathfrak{S}) = h(\mathfrak{F})$. As before, we have

$$Hom_{M_A(\mathfrak{S})}(h(\mathfrak{X}/\mathfrak{S}), h(\mathfrak{Y}/\mathfrak{S})) = Hom_{Cor_{A_{mot}}(\mathfrak{S})}(\mathfrak{X}/\mathfrak{S}, \mathfrak{Y}/\mathfrak{S})$$

We define realization functors from $M_A(\mathfrak{S})$ to the category of G -equivariant polarizable variations of Hodge structures on $S \times_{\sigma} Spec \mathbb{C}$ by

$$H^i(h(f)(m)) = R^{i+2m} f_* \mathbb{Q}(m)$$

$$H(h(f)(m)) = \bigoplus_i H^i(h(f)(m))$$

We can define realizations to G -equivariant ℓ -adic locally constant sheaves in a similar fashion. Given $\mathfrak{F} : \mathfrak{X} \rightarrow \mathfrak{Y}, \mathfrak{G} : \mathfrak{Y} \rightarrow \mathfrak{S}$ in $SPVar_{\mathfrak{S}}$, we say that \mathfrak{G} is motivated by \mathfrak{F} if it lies in the subcategory of $M_A(\mathfrak{S})$ generated from \mathfrak{F} by taking sums, summands, and products. It follows that if \mathfrak{G} is motivated by \mathfrak{F} , then $H(\mathfrak{G})$ lies in the tensor category generated by $H(\mathfrak{F})$ and Tate structures.

Theorem 2.3. *The category $M_A(\mathfrak{S})^{str}$ is a semisimple Tannakian category, and the realization functors gives exact faithful embeddings of this into the Abelian categories of polarizable Hodge structures and locally constant ℓ -adic sheaves.*

We reduce this to a series of lemmas.

Lemma 2.4. *Let R be a finite dimensional algebra over \mathbb{Q} , such that it possesses a trace $\tau : R \rightarrow \mathbb{Q}$ and an algebra involution $a \mapsto a'$ such that the bilinear form $\tau(ab')$ is positive definite. Then R is semisimple.*

Proof. [K, 3.13]. □

Lemma 2.5. *We use the same notation and assumptions as in proposition 2.1, in particular that $\kappa = \mathbb{C}$. Under the inclusion $A_{mot}^*(X/S) \subset H^*(X_s)$, $A_{mot}^*(X/S)$ is invariant under the Hodge involution, as defined in [An, pp. 10-11], with respect to an ample G -invariant line bundle (which exists) on \bar{X} .*

Proof. Pick an ample line bundle \mathcal{L} on \bar{X} , replace it with $\otimes_{g \in G} g^* \mathcal{L}$, and equip \bar{X} and X_s with the associated Kähler metrics. The Hodge involution $*$ is the same as the Hodge star operator (up to a factor and complex conjugation). This is G -equivariant since the metric is invariant. One can check that the Hodge star operator is compatible with restriction of Kähler manifolds. Therefore $H^*(X)^{\pi_1(S)}$, which equals $im(H^*(\bar{X}) \rightarrow H^*(X_s))$ by the theorem of the fixed part [De, 4.2], is stable under $*$, and hence so is $H^*(X)^{\pi_1(\mathfrak{S})}$. The invariance is also true for $A_{mot}^*(X_s)$ by [An, 2.2], and thus also for the intersection of these spaces. □

By comparison, we get the same conclusion in general.

Corollary 2.6. *The lemma holds for ℓ -adic cohomology for any field κ of characteristic zero.*

Proof of theorem 2.3. To prove that $M = M_A(\mathfrak{S})$ is Abelian and semisimple, it is enough by [J, lemma 2] to prove that $End_M(T)$ is semisimple for each motive T . We can assume that $T = h(\mathfrak{X}/\mathfrak{S})$ for some smooth projective morphism of relative dimension d . Then by lemma 2.5

$$End_M(T) = A_{mot}^d(\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}/\mathfrak{S}) \subset H^*(X_s \times X_s)$$

is stable under the involution $a' = *a^t*$, where a^t is the transpose (c.f. [K, 1.3]). With the help of the Hodge index theorem, we see that this algebra satisfies the conditions of lemma 2.4 (see [loc. cit, p. 381]). Therefore it is semisimple.

The functor H is exact, since any additive functor between semisimple categories is exact. The faithfulness follows from proposition 2.1. The tensor structure is induced by fibre product: $h(\mathfrak{X}/\mathfrak{S}) \otimes h(\mathfrak{Y}/\mathfrak{S}) = h(\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}/\mathfrak{S})$. The verification that this is Tannakian is essentially the same argument as in [An, 4.3] and [J, cor. 2]. \square

Given a morphism $\mathfrak{T} \rightarrow \mathfrak{S}$ of stacks satisfying the above assumptions, we have a base change map $M_A(\mathfrak{S})^{str} \rightarrow M_A(\mathfrak{T})^{str}$ for the categories of motives.

Corollary 2.7. *The base change map is always exact. Given a point $s \in S$, the functor $M_A(\mathfrak{S})^{str} \rightarrow M_A(s)^{str} = M_A$ is an exact faithful embedding.*

Proof. As already noted an additive functor between semisimple categories is exact. The composition $M_A(\mathfrak{S})^{str} \rightarrow M_A \rightarrow \mathbb{Q}_{\ell}\text{-vect}$ to the category of ℓ -adic vector spaces is exact and faithful. This also factors as a composition of H and the exact faithful fibre functor from ℓ -adic local systems to $\mathbb{Q}_{\ell}\text{-vect}$. \square

It will be convenient to define a slightly bigger category of relative motives $M_A(\mathfrak{S})$. An object consists of a strict motive in $M_A([U/G])$ for some nonempty G -invariant Zariski open $U \subset S$, such that the underlying ℓ -adic local system extends to S , for some fixed ℓ . It is easily seen to be equivalent to requiring the extendibility of topological local system on $U \times_{\sigma} \text{Spec } \mathbb{C}$ for some σ . In particular, this notion is independent of the choice of ℓ or σ . Given two motives R, Q defined over open sets U and V respectively, let

$$\text{Hom}_{M(\mathfrak{S})}(R, Q) = \text{Hom}_{M(\mathfrak{S})^{str}}(R|_{U \cap V}, Q|_{U \cap V})$$

By a theorem Griffiths [G, 9.5], the Hodge realization functor H extends to this category $M_A(\mathfrak{S})$. The analogue of theorem 2.3 is easily checked. As before, objects in the category $M_A(\mathfrak{S})$ admit a weight grading $T = \bigoplus w^i(T)$ such that $w^i(T)$ maps to the maximal sub variation of Hodge structure of weight i of the realization of T . We note in passing that the construction also works when S is singular. In this case, we define a motive on \mathfrak{S} as a strict motive on a smooth open G -invariant set $U \subset S$ for which the local system extends. The notion of a variation of Hodge structure can be extended in a similar fashion.

3. THE MOTIVE OF A STACK

It will be technically convenient to formally adjoin arbitrary direct sums to $M_A(\kappa) = M_A(\text{Spec } \kappa)$. We do this by working in the category $\text{Ind-}M_A(\kappa)$ of ind-objects of $M_A(\kappa)$. This category is again Abelian and semisimple. Moreover it has arbitrary direct sums:

$$\bigoplus_{i \in I} A_i = \varinjlim_{\text{finite } J} \left(\bigoplus_{j \in J} A_j \right)$$

In this section, we show how to associate an object $h(\mathfrak{X})$ in the derived category $D^+(\text{Ind-}M_A(\kappa))$ for any Artin stack. This would lie in $D^+(M_A(\kappa))$ when \mathfrak{X} has finite type. Note that by semisimplicity, this decomposes as $h(\mathfrak{X}) \cong \bigoplus_i \mathcal{H}^i(h(\mathfrak{X}))[-i]$ where $\mathcal{H}^i(h(\mathfrak{X}))$ is the i th cohomology. So h can be viewed as a graded motive.

Consider a pair (X, D) , where X is a smooth projective variety and D is a normal crossings divisor on X defined over κ . Let D_1, \dots, D_n be the irreducible components of D . For a subset $I \subseteq \{1, \dots, n\}$ define

$$D_I = \bigcap_{i \in I} D_i.$$

Note that $D_\emptyset = X$.

The indexing set I has a natural order inherited from $[n]$ and we set $I_k = I - \{i_k\}$ for $1 \leq k \leq |I|$. Also define

$$D^{(l)} = \coprod_{|I|=l} D_I.$$

The inclusions $D_I \hookrightarrow D_{I_k}$ induce a natural map

$$\delta_k : D^{(i)} \rightarrow D^{(i-1)}.$$

Passing to Andre's category of motives $M_A(\kappa)$ we have maps

$$h(\delta_k) : h(D^{(i-1)}) \rightarrow h(D^{(i)}).$$

Dualizing we obtain

$$h(\delta_k)^* : h(D^{(i)})(-1) \rightarrow h(D^{(i-1)}).$$

We define $h(X, D)$ to be the complex (or its image in the derived category) given by

$$h(D^{(n)})(-n) \rightarrow \cdots \rightarrow h(D^{(0)})$$

where the differentials are the alternating sums of the δ_k 's, and $h(D^{(0)})$ is positioned in degree 0. The following is immediate.

Proposition 3.1. *This construction is contravariantly functorial in the pair (X, D) .*

Fix an embedding of the ground field $\kappa \subset \mathbb{C}$, then we can regard $M_A(\kappa)$ as a subcategory of the category of rational pure Hodge structures $\mathbb{Q}\text{-HS}$ via the Hodge realization functor H . To simplify notation, we will often omit this symbol. This functor is exact and so extends to the derived categories. We will use the same symbol for the derived functor.

There are functors $w^i : \mathbb{Q}\text{-HS} \rightarrow \mathbb{Q}\text{-HS}$ which project onto the weight i piece of the Hodge structure. These are compatible with the previous $w^i : M_A(\kappa) \rightarrow M_A(\kappa)$ in the sense that they commute with H .

Proposition 3.2. *Let (X, D) be as above and let $U = X \setminus D$. We denote by W the weight filtration for the mixed Hodge structure on $H^*(U, \mathbb{Q})$. Then there is a canonical isomorphism*

$$w^j \mathcal{H}^i(h(X, D)) \cong Gr_W^j H^{j+i}(U, \mathbb{Q}),$$

(after realization) where \mathcal{H}^i denote the i th cohomology of the complex.

Proof. [De, 3.2.13] yields a spectral sequence

$$E_1^{-m, k+m} = H^{k-m}(D^{(m)})(-m) \Rightarrow H^k(U)$$

with $E_2 = E_\infty$. Since E_∞ gives the weight graded subquotients of the abutment, and the E_1 complex coincides with $w^* \mathcal{H}^*(h(X, D))$, the result is certainly true

qualitatively. But we need to calculate the precise indices:

$$\begin{aligned} w^j \mathcal{H}^i(h(X, D)) &= \mathcal{H}^i(w^j(h(X, D))) \\ &= \mathcal{H}^i(\dots \rightarrow H^{j-2m}(D^{(m)})(-m) \rightarrow \dots) \\ &= E_2^{i,j} \\ &= Gr_W^j H^{j+i}(U) \end{aligned}$$

In the first line, we use the fact that w^j is exact and thus commutes with \mathcal{H}^i . \square

If we regard $\mathbb{C}_{D^{(i)}}$ as sheaves on X , then we form the complex

$$\mathbb{C}_{D^{(n)}}[-2n] \rightarrow \dots \rightarrow \mathbb{C}_{D^{(0)}}[0]$$

in the derived category, where the differentials are alternating sums of Gysin maps. Let σ^k denote the stupid filtration [De, 1.4.7]. As a corollary of the proof, we obtain:

Corollary 3.3. *There is an isomorphism $\mathcal{H}^i(h(X, D)) \otimes \mathbb{C} \cong H^i(\mathbb{C}_{D^{(\bullet)}}[\bullet])$ under which images under w^\bullet on the left correspond to σ -graded components on the right.*

It will be important to chose a canonical representative for this complex. Via Poincaré residues, we can realize $\mathbb{C}_{D^{(k)}}[-2k] \rightarrow \mathbb{C}_{D^{(k-1)}}[2-2k]$ as the morphism $\Omega_{D^{(k)}}^\bullet[-2k] \rightarrow \Omega_{D^{(k-1)}}^\bullet[2-2k]$ in the sense of derived categories given by

$$\begin{array}{ccccc} & & \text{Cone}(W_k[-k] \rightarrow \Omega_{D^{(k)}}^\bullet[-2k]) & & \\ & \nearrow & \uparrow & & \\ \Omega_{D^{(k)}}^\bullet[-2k] & & W_{k-1}[1-k] & \longrightarrow & \Omega_{D^{(k-1)}}^\bullet[2-2k] \end{array}$$

where $W_\bullet = W_\bullet \Omega_X^\bullet(\log D)$ and the double arrow is a quasi-isomorphism. Then by repeated use of mapping cones (as in [GM, pp 161-162]), we can use these maps to build a complex of sheaves $K_{(X, D)}^\bullet$ on X quasi-isomorphic to $\mathbb{C}_{D^{(\bullet)}}[\bullet]$, which is functorial in the pair (X, D) .

We can construct a mild generalization of the above. Let (X_\bullet, D_\bullet) be a simplicial logarithmic pair defined over κ . Then complexes $h(X_k, D_k)$ fit into a double complex in $M_A(\kappa)$, where the second differentials are alternating sums of face maps (with signs suitably adjusted to anticommute). Let $h(X_\bullet, D_\bullet)$ be the total complex in $C^+(M_A(\kappa))$. This determines an object in $D^+(M_A(\kappa))$ denoted by the same symbol. Similarly, we can build a complex of sheaves $K_{(X, D)}^\bullet$ on X_\bullet .

Theorem 3.4. *Let $U_\bullet = X_\bullet \setminus D_\bullet$. We have*

$$w^j \mathcal{H}^i(h(X_\bullet, D_\bullet)) = Gr_W^{i+j} H^j(U_\bullet, \mathbb{Q}).$$

(after realization).

Proof. We have a spectral sequence of MHS

$$(1) \quad {}_1 E_1^{pq} = H^q(U_p) \Rightarrow H^{p+q}(U_\bullet)$$

by [De, 8.3.4]. After tensoring by \mathbb{C} , this can be constructed as the spectral sequence for total direct image

$$Tot(\mathbb{R}\Gamma(\Omega_{X_0}^\bullet(\log D_0)) \rightarrow \mathbb{R}\Gamma(\Omega_{X_1}^\bullet(\log D_1)) \rightarrow \dots)$$

associated to the filtration by skeleta

$$Tot(\dots 0 \rightarrow \mathbb{R}\Gamma(\Omega_{X_p}^\bullet(\log D_p)) \rightarrow \mathbb{R}\Gamma(\Omega_{X_{p+1}}^\bullet(\log D_{p+1})) \dots)$$

On the other hand, filtering the double complex defining $h(X_\bullet, D_\bullet)$ by skeleta, yields a spectral sequence

$${}_{II}E_1^{pq} = \mathcal{H}^q(h(X_p, D_q)) \Rightarrow \mathcal{H}^{p+q}(h(X_\bullet, D_\bullet))$$

Therefore after applying w^j and using proposition 3.2, we get a spectral sequence

$$E_1^{pq} = Gr_W^j H^{q+j}(U_p) \Rightarrow w^j \mathcal{H}^{p+q}(X_\bullet, D_\bullet)$$

So it suffices to show that this coincides with the spectral sequence resulting from applying Gr_W^j to (1) and shifting. But this follows from the previous discussion, since we can construct a spectral sequence using the skeletal filtration on

$$Tot(K_{(X_\bullet, D_\bullet)})$$

which maps to both ${}_I E_1$ and ${}_{II} E_1$. \square

Corollary 3.5. *For any simplicial algebraic space Y_\bullet over κ , with each Y_n of finite type, we have an object $h(Y_\bullet) \in D^+(M_A(\kappa))$ satisfying*

$$w^j \mathcal{H}^i(h(Y_\bullet)) = Gr_W^{i+j} H^j(Y_\bullet, \mathbb{Q})$$

(after realization).

Proof. By standard arguments (cf. [De, 8.3.6]), we can construct a simplicial logarithmic pair (X_\bullet, D_\bullet) such that $U_\bullet = X_\bullet - D_\bullet$ has the same cohomology as Y_\bullet . Moreover, if (X'_\bullet, D'_\bullet) is another such scheme, we can assume without loss of generality that it factors as

$$\alpha : (X'_\bullet, D'_\bullet) \rightarrow (X_\bullet, D_\bullet)$$

Since this induces an isomorphism of mixed Hodge structures on cohomology, $h(\alpha)$ must be a quasi-isomorphism by the theorem. \square

Corollary 3.6. *Let \mathfrak{X} be a stack of finite type over κ . Then $h(\mathfrak{X}) = h(Y_\bullet)$ gives well define class in $D^+(M_A(\kappa))$, where Y_\bullet is the nerve of any presentation of \mathfrak{X} .*

Proof. This is well defined, since the mixed Hodge structure on cohomology depends only on \mathfrak{X} [Dh, T]. \square

In view of the above results, it makes sense to define

$$(2) \quad h^i(\mathfrak{X}) = \bigoplus_j w^j \mathcal{H}^{i-j}(h(\mathfrak{X}))$$

for a stack or simplicial space. For a smooth projective variety, this agrees with the previous meaning. In general, under Hodge realization, $h^i(\mathfrak{X})$ would map to $Gr_W H^i(\mathfrak{X}) = \bigoplus_j Gr_W^j H^i(\mathfrak{X})$.

We can refine the construction in the following ways:

- (H1) For any stack \mathfrak{X} (locally of finite type as always), we get a class $h(\mathfrak{X}) \in D^+(\text{Ind-}M_A(\kappa))$, such that $h^i(\mathfrak{X})$ defined as in (2) maps to the infinite dimensional Hodge structure $Gr_W H^i(\mathfrak{X})$.

- (H2) If $f : X \rightarrow S$ is a smooth projective morphism, and $D \subset X$ a relative normal crossing divisor, we can define a complex $h((X, D)/S) \in C^+(M_A(S)^{str})$ (and hence a class in its derived category) such that

$$w^j \mathcal{H}^i(h(X, D)/S) = Gr_W^j R^{j+i} g_* \mathbb{Q}$$

(after realization) where $g : U \rightarrow S$ is the restriction to the complement. Or equivalently, $h^i((X, D)/S) = Gr_W R^i g_* \mathbb{Q}$ with the above convention (2). The construction is compatible with base change.

- (H3) When a finite group G acts equivariantly on $f : X \rightarrow S$ and D as above, the previous class descends to an element of $C^+(M_A([S/G])^{str})$.
- (H4) Let $\mathfrak{S} = [S/G]$ be the quotient of a smooth variety by a finite group. Call a morphism $\mathfrak{F} : \mathfrak{X} \rightarrow \mathfrak{S}$ cohomologically locally constant (and finite) if all the direct images $R^i(\mathfrak{F} \times_{\mathfrak{S}} S)_* \mathbb{Q}$ are locally constant (with finite dimensional stalks). Then to any cohomologically locally constant morphism, we can construct motives $h^i(\mathfrak{X}/\mathfrak{S})$ in $\text{Ind-}M_A(\mathfrak{S})$ (or $M_A(\mathfrak{S})$ assuming finiteness) compatible with base change.

Items (H1)-(H3) are straight forward modifications of the previous construction, so the details will be omitted. However, we will say a few words about (H4). Given a cohomologically locally constant finite morphism $\mathfrak{F} : \mathfrak{X} \rightarrow \mathfrak{S}$, we can find a G -equivariant simplicial space $f_{\bullet} : Y_{\bullet} \rightarrow S$, such that $R^i(\mathfrak{F} \times_{\mathfrak{S}} S)_* \mathbb{Q} = R^i f_{\bullet*} \mathbb{Q}$ for all i . We now fix i . Then $R^i f_{\bullet*} \mathbb{Q}$ depends on the $(i+1)$ skeleton which is a finite diagram. By resolution of singularities applied to the generic fibre and descent theory [De], we see that there exists a nonempty G -invariant open set $U \subset S$, and a G -equivariant $(i+1)$ -truncated simplicial relative logarithmic pair $g_{\bullet} : (X_{\bullet}, D_{\bullet}) \rightarrow U$ such that $R^i(\mathfrak{F} \times_{\mathfrak{S}} S)_* \mathbb{Q}|_U = R^i g_{\bullet*} \mathbb{Q}$. Set

$$h^i(\mathfrak{X}/\mathfrak{S}) = \bigoplus_j w^j \mathcal{H}^{i-j}(Tot(h(X_{\bullet}, D_{\bullet})/U)) \in M_A([U/G])^{str}$$

By arguing as in the proof of theorem 3.4, we can see that this maps to $Gr_W R^i(\mathfrak{F} \times_{\mathfrak{S}} S)_* \mathbb{Q}$ under Hodge realization. Hence it is independent of choices and extends to $M_A(\mathfrak{S})$.

4. MOTIVE OF THE MODULI STACK

Throughout this section, G denotes a split semisimple group over κ . Fix an embedding $\kappa \subset \mathbb{C}$. By base change we get a complex group G and stack BG over \mathbb{C} , which will be denoted by the same symbols when no confusion is likely. As noted earlier, the space $|BG|$ associated to BG is the classifying space in the usual sense, and we will usually write BG for both objects to simplify notation.

4.1. Cohomology of the classifying space. We recall the description of cohomology of BG and associated spaces. (The calculations are unchanged if G is replaced by a maximal compact. After doing so, proofs can be found in [B, W].) We have an isomorphism

$$H^*(BG, \mathbb{Q}) \cong H^*(BT, \mathbb{Q})^W = \mathbb{Q}[x_1, \dots, x_n]^W,$$

where $T = \mathbb{G}_m^n$ is a maximal torus with Weyl group W and x_i is the first Chern class of universal line bundle on the i th factor. The right hand side is a polynomial ring in the elementary W -invariant polynomials of the x_i . These are Chern classes of the universal bundle EG . Let $2n_i$ denote the degrees of these Chern classes,

and let $N = \bigoplus N_i$ denote the span of these classes. Since G is semisimple, these numbers are greater than 2. These Chern classes define a map

$$BG \rightarrow \prod K(\mathbb{Z}, 2n_i)$$

to a product of Eilenberg-MacLane spaces which induces a *rational* homotopy equivalence. If we identify G with the based loop space $\Omega BG = Map^*(S^1, BG)$, then we get a rational homotopy equivalence with $\prod K(\mathbb{Z}, 2n_i - 1)$. It follows that $H^*(G, \mathbb{Q})$ is an exterior algebra on $N[1]$ (where $N[i]_n = N_{n+i}$). This can be seen from a different point of view by applying a theorem of Hopf, then $N[1]$ corresponds to the space of primitive elements for the Hopf algebra structure. In more explicit terms, $N[1]$ can be identified with subspace of $H^*(G)$ by taking the image of N under the composition

$$H^*(BG) \xrightarrow{e^*} H^*(G \times S^1) \xrightarrow{f_{s^1}} H^{*-1}(G),$$

of the pullback along evaluation $e : G \times S^1 \rightarrow BG$ and slant product. The loop space ΩG which is homotopic to $\Omega^2 BG$ has $|\pi_1(G)|$ connected components, each rationally equivalent to $\prod K(\mathbb{Z}, 2n_i - 2)$. The cohomology of each component is a symmetric algebra on $N[2]$. These generators can be obtained from N by the above procedure.

Proposition 4.2. *Let G be a split connected semisimple group then $h(BG)$ is quasi-isomorphic to a direct sum of translates of Tate motives.*

Proof. Let T be a maximal torus inside G and W the corresponding Weyl group. The bar construction gives a model of BT as a simplicial scheme T_\bullet for which each T_n is a union of a product of \mathbb{G}_m 's. It follows immediately that $h(BT)$ is a direct sum of translated Tate motives. Since the cohomology ring of BT is a polynomial ring in $\dim T$ variables, it follows that there is a quasi-isomorphism

$$h(BT) \cong \bigoplus_{i=0}^{\infty} \mathbb{Q}(-i)^{r(i)}[-2i]$$

where $r(i) = \binom{\dim T+i-1}{i}$. The Weyl group acts on the right in a way that is compatible with the action on $h(BT)$. The natural map $BT \rightarrow BG$ induces an isomorphism

$$h(BG) \cong \left(\bigoplus_{i=0}^{\infty} \mathbb{Q}(-i)^{r(i)}[-2i] \right)^W$$

as $H^*(BG) = H^*(BT)^W$. □

4.3. Cohomology of \mathfrak{Bun}_G . Fix a smooth projective curve C of genus g over \mathbb{C} . Atiyah and Bott [AB] described the cohomology ring of the mapping space $Map(C, BG)$ and Teleman [T] showed that this space can be identified with $\mathfrak{Bun}_G = \mathfrak{Bun}_{G,C}$. We review these results in a form that is convenient for us.

In general, \mathfrak{Bun}_G has $|\pi_1(G)|$ connected components, Let \mathfrak{Bun}_G^c denote one of these. The universal bundle over $C \times \mathfrak{Bun}_G^c$ produces a classifying morphism to BG . Hence there is a pullback map

$$H^*(BG) \rightarrow H^*(C) \otimes H^*(\mathfrak{Bun}_G^c).$$

This can be transposed to obtain

$$\tau_i : H^*(BG) \otimes H^i(C)^* \rightarrow H^{*-i}(\mathfrak{Bun}_G^c).$$

Set

$$\alpha = \tau_0 : H^*(BG) \rightarrow H^*(\mathfrak{Bun}_G^c).$$

The maps τ_1 and τ_2 induce algebra homomorphisms

$$\begin{aligned} \beta : \bigwedge(N[-1] \otimes H^1(C)^*) &\rightarrow H^*(\mathfrak{Bun}_G^c). \\ \gamma : Sym(N[-2](1)) &\rightarrow H^*(\mathfrak{Bun}_G^c). \end{aligned}$$

Theorem 4.4 (Atiyah-Bott, Teleman). $\alpha \otimes \beta \otimes \gamma$ is an isomorphism of mixed Hodge structures.

Proof. This is essentially contained in [AB, section 2] and [T, p. 24], but we outline the main points since some details are only implicit. Since the map is a morphism of mixed Hodge structures, it suffices to prove that it is an isomorphism of vector spaces. The Poincaré series of the domain of $\alpha \otimes \beta \otimes \gamma$ is easily computed to obtain

$$\prod_i \frac{(1+t^{2n_i-1})^{2g}}{(1-t^{2n_i})(1-t^{2n_i-2})}$$

We can check that we get the same series for each \mathfrak{Bun}_G^c by using Thom's theorem (cf [AB, pp 540-541] and [Th]). Note in particular, that Thom shows that the Poincaré polynomial is independent of the choice of connected component. Thus the Poincaré series for \mathfrak{Bun}_G is $|\pi_1(G)|$ times the above series.

We have a cofibration

$$\bigvee_{2g} S^1 \rightarrow C \rightarrow S^2$$

which gives a fibration of base point preserving mapping spaces

$$\Omega^2 BG = Map^*(S^2, BG) \xrightarrow{c} Map^*(C, BG) \xrightarrow{b} Map^*(\bigvee_{2g} S^1, BG) \sim \prod_{2g} G$$

where \sim denotes homotopy equivalence. This together with the fibration

$$Map^*(C, BG) \rightarrow Map(C, BG) \sim \mathfrak{Bun}_G \xrightarrow{a} BG$$

yields a “3 dimensional spectral sequence”

$$E_2^{pqr} = H^p(BG) \otimes H^q(G^{2g}) \otimes H^r(\Omega G) \Rightarrow H^{p+q+r}(\mathfrak{Bun}_G)$$

Note that the sum of terms on the left is just a sum of $|\pi_1(G)|$ copies the domain of $\alpha \otimes \beta \otimes \gamma$. So the equality of Poincaré series forces $E_2 = E_\infty$, and thus we have an isomorphism. \square

Corollary 4.5. The mixed Hodge structure on \mathfrak{Bun}_G is pure, i.e. a direct sum of pure Hodge structures.

4.6. Motive of the moduli stack. Let $C \rightarrow S$ be a family of genus g curves. Then we have a S -stacks $BG \times S$ and $\mathfrak{Bun}_G = \mathfrak{Bun}_{G,C/S}$. We can apply theorem 4.4 fibrewise to conclude that \mathfrak{Bun}_G/S is cohomologically locally constant and finite. Consequently the motive $h(\mathfrak{Bun}_G/S) \in M_A(S)$ is defined. The universal bundle gives a morphism

$$h^*(BG \times S/S) \rightarrow h^*(C/S) \otimes h^*(\mathfrak{Bun}_G/S)$$

as above. It is also clear after passing to the simplicial model, that we can form the transpose

$$\tau_i : h^*(BG \times S/S) \otimes h^i(C/S)^* \rightarrow h^{*-i}(\mathfrak{Bun}_G/S)$$

and maps α, β, γ as we did earlier (where we use the tensor structure on $M_A(S)$ to define exterior and symmetric powers). With this set up, we get as a corollary to theorem 4.4

Corollary 4.7. $\alpha \otimes \beta \otimes \gamma$ is an isomorphism of motives

When finite group Γ acts on S and the family C/S , the above isomorphism descends to $M_A([S/\Gamma])$. Applying this to the universal curve $\pi : \mathfrak{C}_g \rightarrow \mathfrak{M}_g$ (extended to $\text{Spec } \kappa$) yields

Theorem 4.8. The motive of $\mathfrak{Bun}_{G,\mathfrak{C}_g}/\mathfrak{M}_g$ is isomorphic to

$$h^*(BG \times \mathfrak{M}_g/\mathfrak{M}_g) \otimes h^*(\bigwedge(N[-1] \otimes h^1(\mathfrak{C}_g/M_g)^*)) \otimes h^*(\text{Sym}(N[-2](1)))$$

Corollary 4.9. The same isomorphism holds for the variation of Hodge structure associated to $\mathfrak{Bun}_{G,\mathfrak{C}_g}/\mathfrak{M}_g$, and in particular for its monodromy representation.

Recall that the Torelli group is the kernel of the monodromy representation

$$\Gamma_g \rightarrow Sp(2g, \mathbb{Z})$$

associated to $R^1\pi_*\mathbb{Z}$. As a subcorollary, we see that the action of Torelli group on cohomology of \mathfrak{Bun}_G is trivial.

5. COMPARISON WITH THE MODULI SPACE

Fix a reductive group G and a smooth projective genus g curve C , both defined over κ .

5.1. (Semi)-Stability for \mathcal{G} -bundles. By a principal G -bundle or simply G -bundle over C , we will mean a scheme $P \rightarrow C$, with a right G -action, which is étale locally a product. To every G -bundle P over a curve C , we can form the smooth affine group scheme $\mathcal{G} = \text{Aut}(P) = P \times_{G, \text{Ad}} G$. This is reductive since G is. \mathcal{G} will carry all the information we need, and it is technically more convenient to work with it. We define the *degree* of a smooth affine group scheme \mathcal{G} over C to be the degree of the vector bundle $\text{Lie}(\mathcal{G})$ over C . It is denoted by $\deg \mathcal{G}$

The following fact is very useful :

Lemma 5.2. Let \mathcal{G} be a reductive group scheme over C . There is a finite étale cover $f : Y \rightarrow C$ such that $f^*\mathcal{G}$ is an inner form.

Proof. We make use of the notations of [DG]. Let \mathcal{G}_0 be the constant reductive group scheme over C having the same type as \mathcal{G} . Being an inner form means that the scheme $\text{Isomext}(\mathcal{G}, \mathcal{G}_0)$ has a section over C . By [DG, XXIV, theorem 1.3] and by [DG, XXII, corollary 2.3] G is quasi-isotrivial and hence so is $\text{Isomext}(\mathcal{G}, \mathcal{G}_0)$. This implies by [DG, X, corollay 5.4] that $\text{Isomext}(\mathcal{G}, \mathcal{G}_0)$ is étale and finite over C . So we take Y to be one of these components and the section is the tautological section. \square

Corollary 5.3. If \mathcal{G} is a reductive group scheme over C then $\deg \mathcal{G} = 0$.

Proof. By the above we may assume that \mathcal{G} is an inner form. If \mathcal{G}_0 is the constant reductive group scheme of the same type as \mathcal{G} then the adjoint action of \mathcal{G}_0 on its Lie algebra factors through $\text{SL}(\text{Lie}(\mathcal{G}_0))$. \square

Definition 5.4 (Behrend). A reductive group scheme \mathcal{G}/C is said to be stable (resp. semistable) if for every proper parabolic subgroup \mathcal{P} of \mathcal{G} we have $\deg \mathcal{P} < 0$ (resp. $\deg \mathcal{P} \leq 0$). The degree of the largest parabolic subgroup of \mathcal{G} is called the degree of instability of \mathcal{G} and denoted $\deg_i \mathcal{G}$.

Let E be a \mathcal{G} -bundle. We denote by ${}^E\mathcal{G}$ the associated inner form

$${}^E\mathcal{G} = E \times_{\mathcal{G}, \text{Ad}} \mathcal{G}.$$

We say that E is (semi)-stable if ${}^E\mathcal{G}$ is and define $\deg_i \mathcal{G} = \deg_i {}^E\mathcal{G}$.

In the case of a constant group scheme $\mathcal{G} = G \times_k C$ this definition of (semi)-stability is equivalent to the usual one in [Ra2]. This equivalence is by the remarks in the last paragraph on page 304 of [Be2]. Basically there is a bijection between parabolic subgroups of ${}^E\mathcal{G}$ and reductions of structure group of E to a parabolic subgroup of G . This bijection underlies the equivalence of the two definitions.

5.5. The Bounds. Denote by $X(\mathcal{G}) = \text{Hom}(\mathcal{G}, \mathbb{G}_m)$ the group of characters of \mathcal{G} and by $X(\mathcal{G})^\vee = \text{Hom}(X(\mathcal{G}), \mathbb{Z})$ its dual.

Given a \mathcal{G} -bundle P its degree is defined to be the element of $X(\mathcal{G})^\vee$ defined by

$$\chi \mapsto \deg(P \times_{\mathcal{G}} \mathbb{G}_m),$$

where $\chi \in X(\mathcal{G})$. Notice that $P \times_{\mathcal{G}} \mathbb{G}_m$ can be viewed as a line bundle on our curve. For $\alpha \in X(\mathcal{G})^\vee$ we denote by $\mathfrak{Bun}_{\mathcal{G}}^\alpha$ the component of $\mathfrak{Bun}_{\mathcal{G}}$ consisting parameterizing bundles of degree α . It is a union of connected components of $\mathfrak{Bun}_{\mathcal{G}}$.

Recall that the stack $\mathfrak{Bun}_{\mathcal{G}}^\alpha$ has dimension $\dim_C \mathcal{G}(g - 1)$ where \dim_C is the relative dimension over C . For a parabolic subgroup $\mathcal{P} \subseteq \mathcal{G}$ and a character $\beta \in X(\mathcal{P})^\vee$ such that the universal bundle on $\mathfrak{Bun}_{\mathcal{P}}^\beta$ has non-negative degree we have

$$\dim \mathfrak{Bun}_{\mathcal{P}}^\beta \leq \dim_C \mathcal{P}(g - 1).$$

This follows directly from [Be1, proposition 8.1.7]. See also [BD, proposition 5.8]. Let \mathfrak{C} be the complement of the stable locus in $\mathfrak{Bun}_{\mathcal{G}}^\alpha$. Observe that, by the definition of stability, \mathfrak{C} is the union of the images

$$\mathfrak{Bun}_{\mathcal{P}}^\beta \rightarrow \mathfrak{Bun}_{\mathcal{G}}^\alpha$$

where the universal bundle on $\mathfrak{Bun}_{\mathcal{P}}^\beta$ has non-negative degree and the image of β under

$$X(\mathcal{P})^\vee \rightarrow X(\mathcal{G})^\vee$$

is α . It follows from the above that \mathfrak{C} has codimension at least

$$\begin{aligned} d_{\mathcal{G}} &= \min(\dim_C \mathcal{G} - \dim_C \mathcal{P})(g - 1) \\ &= \min(\dim_C R_u(\mathcal{P}))(g - 1). \end{aligned}$$

In the above equation, “min” runs over all proper parabolic subgroups of \mathcal{G} .

In the case where $\mathcal{G} = C \times_k G$ with G being a split reductive group, one can interpret the above bound on the codimension in terms of the root datum of G .

First let us recall some facts about parabolic subgroups $P \subseteq G$. Each such subgroup contains a Borel subgroup B and a maximal torus $T \subseteq B$. This data determines a set of roots $R \subseteq X(G)(T) \otimes \mathbb{Q}$, a set of positive roots R_+ and a basis Δ of R .

Let $I \subseteq \Delta$ and let R_I be the set of roots that are linear combinations of roots in I . Let W_I be the subgroup of the Weyl group generated by the reflections S_α with $\alpha \in I$. If

$$P_I = \cup_{w \in W_I} B\tilde{w}B \quad (\tilde{w} \in N_G(T) \text{ a representative for } w)$$

then P_I is a parabolic subgroup of G , see [Sp, Theorem 8.4.3]. Furthermore, there is a $J \subseteq \Delta$ so that $P_J = P$. If $\Lambda_I = R_+ \setminus R_I$ then

$$\dim R_u(P_I) = |\Lambda_I|.$$

Putting this all together it follows that the bound on the codimension of C above is the same as

$$d_G = \min_I(|\Lambda_I|(g-1))$$

where I runs over all sets of the form $I = \Delta \setminus \{\alpha\}$ for some root $\alpha \in \Delta$.

We will now study this minimum for the standard families of Chevalley groups A_n, B_n, C_n and D_n .

Proposition 5.6. *The minimum values for d_G for the standard families of Chevalley groups A_n, B_n, C_n and D_n are given by the following table*

G	A_n	$B_n \ (n \geq 2)$	$C_n \ (n \geq 2)$	$D_n \ (n \geq 3)$
d_G	$n(g-1)$	$2(n-1)(g-1)$	$2(n-1)((g-1)$	$2(n-1)(g-1)$

Proof. We do this case by case.

A_n This root system can be identified with a subset of the hyperplane $H : \sum x_i = 0$ in \mathbb{R}^{n+1} . If we write e_i for the standard basis of \mathbb{R}^{n+1} then the roots are $e_i - e_j$, for i, j distinct. We have the basis

$$\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n\}.$$

Hence

$$R_+ = \{e_i - e_j \mid i < j\}.$$

A straightforward calculation shows that $d_G = n(g-1)$ in this case. Note that this agrees with the bound in [Dh] as SL_n is of type A_{n-1} .

B_n This root system can be identified with the subset of \mathbb{R}^n consisting of

$$\pm e_i \quad \text{and} \quad \pm e_i \pm e_j \quad (i \neq j).$$

We can take

$$\Delta = \{\alpha_1, \dots, \alpha_n\}$$

where $\alpha_i = e_i - e_{i+1}$ for $i < n$ and $\alpha_n = e_n$. The positive roots are then :

$$\begin{aligned} \sum_{k=i}^n \alpha_k, \quad &\text{for } 1 \leq i \leq n \\ \sum_{i \leq k < j} \alpha_k, \quad &\text{for } 1 \leq i < j \leq n \\ \sum_{i \leq k < j} \alpha_k + 2(\sum_{j \leq k \leq n} \alpha_k), \quad &\text{for } 1 \leq i < j \leq n \end{aligned}$$

Let $I = \Delta \setminus \{\alpha_m\}$. The dimension of the unipotent radical of P_I is then

$$m + m(n-m) + \frac{m}{2}(2n-m-1).$$

The smallest this can be is $2n-2$, noting that $n \geq 2$.

C_n This root system can be identified with the subset of \mathbb{R}^n consisting of

$$\pm 2e_i \quad \pm e_i \pm e_j \quad i \neq j$$

We can take

$$\Delta = \{\alpha_1, \dots, \alpha_n\}$$

where $\alpha_i = e_i - e_{i+1}$ for $i < n$ and $\alpha_n = 2e_n$. The positive roots are then :

$$\begin{aligned} \sum_{i \leq k < j} \alpha_k, &\quad \text{for } 1 \leq i < j \leq n \\ \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < n} \alpha_k + \alpha_n &\quad \text{for } 1 \leq i < j \leq n \\ 2 \sum_{i \leq k < n} \alpha_k + \alpha_n &\quad \text{for } 1 \leq i \leq n \end{aligned}$$

Let $I = \Delta \setminus \{\alpha_m\}$. The dimension of the unipotent radical of P_I is then

$$m(n-m) + \frac{m}{2}(2n-m-1) + m.$$

The smallest this can be is $2n-2$, noting that $n \geq 2$.

D_n This root system can be identified with the subset of \mathbb{R}^n consisting of

$$\pm e_i \pm e_j \quad i \neq j$$

We can take

$$\Delta = \{\alpha_1, \dots, \alpha_n\}$$

where $\alpha_i = e_i - e_{i+1}$ for $i < n$ and $\alpha_n = e_{n-1} + e_n$. The positive roots are then :

$$\begin{aligned} \sum_{i < k < j} \alpha_k, &\quad \text{for } 1 \leq i < j \leq n \\ \sum_{i \leq k \leq n} \alpha_k &\quad \text{for } 1 \leq i < n \\ \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq l-1} \alpha_k + \alpha_{l-1} + \alpha_l &\quad \text{for } 1 \leq i < j < n \end{aligned}$$

Let $I = \Delta \setminus \{\alpha_m\}$. The dimension of the unipotent radical of P_I is then

$$(m-1)(n-m) + m + \frac{m}{2}(2n-m-3).$$

The smallest this can be is $2n-2$, noting that $n \geq 3$. \square

5.7. The moduli space. Let G be a split semisimple group over κ . In this case the character group $X(C \times_{\kappa} G)$ is trivial. Let S be a scheme over κ with an action of an algebraic group K . Let $f : C \rightarrow S$ be a smooth projective curve over S for which the action of K lifts. Denote by \mathfrak{Bun}_G^s the open substack of $\mathfrak{Bun}_G = \mathfrak{Bun}_{G,C/S}$ parameterizing stable bundles. Ramanathan [Ra2] had originally constructed a coarse moduli space Bun_G^s for \mathfrak{Bun}_G^s , when $S = \mathrm{Spec} \kappa$. We will review its construction in this relative setting, following [Sc].

Theorem 5.8.

- (i) There is a coarse moduli scheme Bun_G^s for the stack \mathfrak{Bun}_G^s .
- (ii) The action of K lifts to Bun_G^s and the natural map $\mathfrak{Bun}_G^s \rightarrow \mathrm{Bun}_G^s$ is equivariant for the K -action.

Proof. Fix a faithful representation

$$\rho : G \rightarrow \mathrm{GL}(V).$$

To give a principal G -bundle on a scheme X is the same as giving a $\mathrm{GL}(V)$ -bundle E plus a reduction of structure group of E to G , in other words a section of E/G . If \mathcal{E} is the vector bundle associated to E , then the reduction can be encoded as a homomorphism of sheaves of algebras

$$\sigma : \mathrm{Sym}^*(\mathcal{E}^\vee \otimes V)^G \rightarrow \mathcal{O}_X.$$

such that the induced section of

$$\underline{\mathrm{Hom}}(V \otimes \mathcal{O}_X, \mathcal{E})/G$$

lifts (locally) to a section of $\underline{\mathrm{Isom}}(V \otimes \mathcal{O}_X, \mathcal{E}) \cong E$.

Note that if a reduction of structure group exists then \mathcal{E} must have degree 0. Fix a relatively ample divisor D on C and set $P(t) = r \deg Dt + r(1-g)$ where $r = \dim V$. The collection of vector bundles \mathcal{E} with Hilbert polynomial P that admit reductions to stable principal G -bundles is a bounded family, see [Sc, 3.2]. So we can find an N so that for any $n \geq N$ and any bundle in the family

$$R^i f_*(\mathcal{E}(nD)) = 0 \quad \text{and} \quad \mathcal{E}(nD) \text{ is generated by global sections.}$$

Let W be a vector space of dimension $P(n)$. Then our bounded family is parameterized by an open subscheme Q of the Quot scheme over S of quotients of $\mathcal{O}_X(-ND) \otimes W$ with Hilbert polynomial P . We have a universal quotient

$$W \otimes p^* \mathcal{O}_X(-ND) \rightarrow \mathcal{Q} \rightarrow 0.$$

As $Q \times X$ is quasi-compact and G reductive, the algebra $\mathrm{Sym}^*(V \otimes \mathcal{Q})^G$ is generated by elements of degree at most k , for some k . Given $q \in Q$, it then follows that a reduction for the corresponding vector bundle

$$\sigma : \mathrm{Sym}^*(V \otimes \mathcal{Q}|_{q \times_S X})^G \rightarrow \mathcal{O}_X$$

is determined by a section of a finite dimensional affine space

$$\Sigma = \underline{\mathrm{Spec}}(\mathrm{Sym}_{\mathcal{O}_S}^* (\bigoplus_{i=0}^k \underline{\mathrm{Hom}}(\mathrm{Sym}^i(V \otimes W)^G \otimes \mathcal{O}_S, f_*(\mathcal{O}(iND)))^\vee)).$$

The set of all possible such pairs $(\sigma, q) \in \Sigma \times Q$ coming from an algebra homomorphism

$$\sigma : \mathrm{Sym}^*(V \otimes \mathcal{Q}|_{q \times_S X})^G \rightarrow \mathcal{O}_X$$

forms a closed subscheme Λ . The action of the algebraic group K lifts to the Quot scheme and preserves the subscheme Λ and this action also commutes with the $\mathrm{GL}(W)$ action. The subset $U \subset \Lambda$ parameterizing stable bundles is open and corresponds to the stable vector bundle locus in a linearization of the action of $\mathrm{GL}(W)$. For this we again refer the reader to [Sc, pg. 1199]. In particular, we can form both the GIT quotient $\mathfrak{Bun}_G^s = U//G$ and the stack theoretic quotient $\mathfrak{Bun}_G^s = [U/G]$. The above discussion shows that the natural morphism

$$\mathfrak{Bun}_G^s \rightarrow \mathrm{Bun}_G^s$$

is K -equivariant. □

We will call the quotient stack $[\mathrm{Bun}_G^s/K]$ the coarse moduli space over $[S/K]$. In particular, this construction yields the coarse moduli space $\mathfrak{Bun}_{G,\mathfrak{C}_g}/\mathfrak{M}_g$ over the universal curve.

Let d be the codimension of the closed complement of \mathfrak{Bun}_G^s in \mathfrak{Bun}_G . For the statement below, we can either take rational Betti cohomology with respect to some embedding $\kappa \subset \mathbb{C}$, or étale cohomology with \mathbb{Q}_ℓ -coefficients.

Proposition 5.9. *Then*

- (i) *The natural map $\mathfrak{Bun}_G^s \rightarrow \mathrm{Bun}_G^s$ induces an isomorphism on rational cohomology in all degrees.*
- (ii) *The inclusion $\mathfrak{Bun}_G^s \rightarrow \mathfrak{Bun}_G$ induces an isomorphism on rational cohomology in degrees smaller than $i < 2d$.*

Proof. The fibre of map $\mathfrak{Bun}_G^s \rightarrow \mathrm{Bun}_G^s$ over a stable bundle P , can be identified with $B\Gamma(\mathrm{Aut}(P))$. Where the group $\Gamma(\mathrm{Aut}(P))$ of global automorphisms of P is finite [Ra1, prop 3.2]. Since the fibres have no higher rational cohomology, the first assertion follows by the Leray spectral sequence. The second assertion can be deduced from the Gysin sequence. \square

Note that d has been calculated in proposition 5.6 for the simple families A_n, B_n, C_n and D_n . We will continue using the notations introduced in subsection 4.6.

Corollary 5.10. *Let $i < 2d$. Then we have an isomorphism of motives*

$$h^i(\mathfrak{Bun}_{G,\mathfrak{C}_g}/\mathfrak{M}_g) \cong h^i(\mathrm{Bun}_{G,\mathfrak{C}_g}^s/\mathfrak{M}_g)$$

Corollary 5.11. *Suppose $i < 2d$, then*

- (1) *The motive $h^i(\mathrm{Bun}_{G,\mathfrak{C}_g}^s/\mathfrak{M}_g)$ is contained in the Tannakian subcategory generated by $\pi : \mathfrak{C}_g \rightarrow \mathfrak{M}_g$.*
- (2) *When $\kappa = \mathbb{C}$, the variation of mixed Hodge structure associated to the i th cohomology of $\mathrm{Bun}_{G,\mathfrak{C}_g}^s/\mathfrak{M}_g$ is pure and lies in the Tannakian subcategory generated by $R^1\pi_*\mathbb{Q}$*

A similar conclusion can be made about the motives of the moduli space of vector bundles of *coprime* rank and degree for all i [Ar2]. However, the above corollary cannot be extended to all i , because Cappell, Lee and Miller [CLM] have shown that the Torelli group would act nontrivially on $H^i(\mathrm{Bun}_{SL_2}^s)$ for most i not less than $\dim \mathrm{Bun}_{SL_2}^s$. In this situation, we expect the variation to be genuinely mixed. In fact, we make the following conjecture:

Conjecture 5.12. *The variation of Hodge structure associated to $\mathrm{Gr}_W H^i(\mathrm{Bun}_{G,C}^s)$ lies in the Tannakian subcategory generated by $R^1\pi_*\mathbb{Q}$ for all i . In particular, the Torelli group acts trivially on this space.*

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